Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 251 (2007) 772–791
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Gelfand transforms of polyradial Schwartz functions on the Heisenberg group

Francesca Astengo [a], Bianca Di Blasio [b], Fulvio Ricci [c,*]

[a] *Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy*
[b] *Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 53, 20125 Milano, Italy*
[c] *Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy*

Received 4 January 2007; accepted 25 June 2007

Available online 10 August 2007

Communicated by L. Gross

Abstract

We prove that the Gelfand transform is a topological isomorphism between the space of polyradial Schwartz functions on the Heisenberg group and the space of Schwartz functions on the Heisenberg brush. We obtain analogous results for radial Schwartz functions on Heisenberg type groups.

Keywords: Heisenberg group; Gelfand pairs; Schwartz functions

1. Introduction

The image of the Schwartz class on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$ under the group Fourier transform has been studied in [6]. Particular attention has been given in Geller's work and in the subsequent literature to special subclasses of the Schwartz class, whose functions commute under convolution and are characterized by invariance under a compact group K of automorphisms of $\mathbb{H}_n$ (see for example [2,4,8,10,12,15]).

Under the usual parametrization of the elements of $\mathbb{H}_n$ by pairs $(z,t) \in \mathbb{C}^n \times \mathbb{R}$, the two most notable examples are the class of "radial" functions, invariant under $K = \mathrm{U}(n)$, and therefore depending only on t and $|z|$, and the class of "polyradial" functions, invariant under a maximal

* Corresponding author.
E-mail addresses: astengo@dima.unige.it (F. Astengo), bianca.diblasio@unimib.it (B. Di Blasio), fricci@sns.it (F. Ricci).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.06.010

torus $\mathbb{T}^n \subset \mathrm{U}(n)$, hence depending on t and on the absolute values $|z_1|, \dots, |z_n|$ of the components of z with respect to an orthonormal basis of $\mathbb{C}^n$ diagonalizing the action of $\mathbb{T}^n$.

If $\mathcal{S}_K(\mathbb{H}_n)$ denotes the space of K-invariant Schwartz functions on $\mathbb{H}_n$, the group Fourier transform of its elements can be equivalently described as the Gelfand transform $\mathcal{F}_K$ relative to the commutative Banach algebra $L^1_K(\mathbb{H}_n)$ of K-invariant integrable functions.

There is a general understanding of the Gelfand spectrum Σ_K of $L^1_K(\mathbb{H}_n)$ and of its topology, as well as a general formula (explicit up to an integration over K) for the bounded K-spherical functions, defining its multiplicative characters.

When $K = \mathrm{U}(n)$, Σ_K is the "Heisenberg fan"

$$\Sigma_{\mathrm{U}(n)} \cong F_n = \big\{\big(\lambda, |\lambda|(2d+n)\big)\colon \lambda \in \mathbb{R}\setminus\{0\},\ d \in \mathbb{N}\big\} \cup \big\{(0,\xi)\colon \xi \geqslant 0\big\},$$

with the induced topology from $\mathbb{R}^2$.

When $K = \mathbb{T}^n$, Σ_K is the "Heisenberg brush"

$$\begin{aligned}\Sigma_{\mathbb{T}^n} \cong B_n = \big\{\big(\lambda, |\lambda|(2d_1+1), \dots, |\lambda|(2d_n+1)\big)\colon \lambda \in \mathbb{R}\setminus\{0\},\ d_j \in \mathbb{N}\big\}\\ \cup \big\{(0, \xi_1, \dots, \xi_n)\colon \xi_1, \dots, \xi_n \geqslant 0\big\},\end{aligned}$$

with the induced topology from $\mathbb{R}^{n+1}$.

For a closed subset E of $\mathbb{R}^m$, denote by $\mathcal{S}(E)$ the space of restrictions to E of Schwartz functions in $\mathbb{R}^m$, endowed with the quotient topology of $\mathcal{S}(\mathbb{R}^m)/\{f\colon f_{|E} = 0\}$. We are interested in establishing that, in the two cases $K = \mathrm{U}(n)$ and $K = \mathbb{T}^n$ and under the identification of Σ_K with F_n and B_n, respectively, the Gelfand transform $\mathcal{F}_K$ induces an isomorphism between $\mathcal{S}_K(\mathbb{H}_n)$ and $\mathcal{S}(\Sigma_K)$. This is easier and more natural than the previous results; indeed, the characterization of [6] is in terms of an asymptotic expansion while that of [2] is in terms of suitable differential–difference operators.

It must be noted that the Gelfand spectrum of a commutative Banach algebra is an abstract topological object, whereas our statement depends on the concrete immersion of Σ_K in some Euclidean space. On the other hand, the immersion of Σ_K as F_n (or B_n) appears to be a very natural one for the following reason.

The bounded K-spherical functions, i.e., the elements of Σ_K, are characterized as the bounded joint eigenfunctions on $\mathbb{H}_n$ of all differential operators on $\mathbb{H}_n$ that commute with left translations and with the action of K, normalized in the L^∞-norm. Call A_K the algebra of such differential operators. If $\{D_1, \dots, D_m\}$ is a set of generators of A_K, we can associate to every bounded K-spherical function the m-tuple $(\mu_1, \dots, \mu_m) \in \mathbb{C}^m$ of its eigenvalues with respect to these generators. It is known [1] (see also [5] for general Gelfand pairs on Lie groups) that this set of m-tuples is a homeomorphic image of Σ_K in $\mathbb{C}^m$.

When $K = \mathrm{U}(n)$, there are two natural, algebraically independent generators of A_K, the imaginary central element $i^{-1}T = i^{-1}d/dt$, and the sub-Laplacian $L = -2\sum_{j=1}^n (Z_j\bar{Z}_j + \bar{Z}_j Z_j)$, where Z_j is the left-invariant complex vector field equal to $\partial/\partial z_j$ at the origin, and $\bar{Z}_j$ is its complex conjugate. The Heisenberg fan consists precisely of all pairs of eigenvalues of bounded K-spherical functions with respect to these generators.

In a similar way, when $K = \mathbb{T}^n$, there are $n+1$ natural generators of A_K, namely $i^{-1}T$ again, and the "partial sub-Laplacians" $L_j = -2(Z_j\bar{Z}_j + \bar{Z}_j Z_j)$. As before, the Heisenberg brush is the set of $(n+1)$-tuples of eigenvalues of bounded K-spherical functions.

The main new result in this paper is the following "extension theorem."

Theorem 1.1. *Let $K = \mathbb{T}^n$. Given f in $\mathcal{S}_K(\mathbb{H}_n)$, its Gelfand transform $\mathcal{F}_K f$ admits a Schwartz extension to $\mathbb{R}^{n+1}$. More precisely, for every Schwartz norm $\| \ \|_{(N)}$, there are a constant C_N and another Schwartz norm $\| \ \|_{(N')}$ such that $\mathcal{F}_K f$ extends to a function Φ_N in $\mathcal{S}(\mathbb{R}^{n+1})$ with $\|\Phi_N\|_{(N)} \leqslant C_N \|f\|_{(N')}$. Similarly for $K = \mathrm{U}(n)$, with $\mathbb{R}^{n+1}$ replaced by $\mathbb{R}^2$.*

The inclusion $\mathcal{S}(B_n) \subset \mathcal{F}_{\mathbb{T}^n}(\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n))$ is proved in [15], as an extension of the results in [7, 12] for spectral multiplier of sub-Laplacians. Putting these two facts together and by the Open Mapping Theorem [14], we have the desired identification.

Corollary 1.2. *The Gelfand transform $\mathcal{F}_{\mathbb{T}^n}$ is a topological isomorphism between $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ and $\mathcal{S}(B_n)$, and $\mathcal{F}_{\mathrm{U}(n)}$ is a topological isomorphism between $\mathcal{S}_{\mathrm{U}(n)}(\mathbb{H}_n)$ and $\mathcal{S}(F_n)$.*

This can be stated in terms of functional calculus of left- and K-invariant differential operators as follows.

Corollary 1.3. *Let $A : \mathcal{S}(\mathbb{H}_n) \to \mathcal{S}'(\mathbb{H}_n)$ be a continuous linear operator commuting with left translations and with the action of $\mathbb{T}^n$, respectively $\mathrm{U}(n)$. Then A has a Schwartz convolution kernel if and only if there is m in $\mathcal{S}(\mathbb{R}^{n+1})$ such that $A = m(i^{-1}T, L_1, \dots, L_n)$, respectively m in $\mathcal{S}(\mathbb{R}^2)$ such that $A = m(i^{-1}T, L)$.*

The same kind of results turns out to hold also in a slightly different situation, with $\mathbb{H}_n$ replaced by a group N of Heisenberg type. In this case, the notion of K-invariance must be replaced by "radiality" on the orthogonal complement of the center. The functions we consider are the $\mathfrak{v}$-radial functions f, which (in the exponential coordinates (X, Z) on N, with $Z \in \mathfrak{z}$, the center of the Lie algebra, and $X \in \mathfrak{v} = \mathfrak{z}^\perp$) depend only on $|X|$ and Z. It is known [3] that $\mathfrak{v}$-radial L^1-functions form a commutative algebra and, as before, there is a natural embedding of its Gelfand spectrum in $\mathbb{R}^{m+1}$, where m is the dimension of $\mathfrak{z}$, as

$$\Sigma_{\mathrm{rad}} = \bigl\{\bigl(\lambda, |\lambda|(2d+n)\bigr) \colon \lambda \in \mathbb{R}^m \setminus \{0\},\ d \in \mathbb{N}\bigr\} \cup \bigl\{(0, \dots, 0, \xi) \colon \xi \geqslant 0\bigr\}.$$

The Gelfand transform then establishes a topological isomorphism between the space $\mathcal{S}_{\mathrm{rad}}(N)$ of $\mathfrak{v}$-radial Schwartz functions and $\mathcal{S}(\Sigma_{\mathrm{rad}})$.

Our paper is structured as follows. We give detailed proofs only for the case where $K = \mathbb{T}^n$ and then indicate the modifications required to treat Heisenberg type groups. This last case also includes the case $N = \mathbb{H}_n$ and $K = \mathrm{U}(n)$. In Section 2 we introduce the basic notions on the Heisenberg group and recall the most relevant facts in the study of the Gelfand transform for polyradial functions. Sections 3 and 4 contain the proof of Theorem 1.1 in the polyradial case; indeed, in Section 3 we extend the Gelfand transform of the polyradial Schwartz function f to a C^k function on $\mathbb{R}^{n+1}$; in Section 4 we adapt some results from [11,16] which let us conclude the proof of Theorem 1.1. Section 5 deals with $\mathfrak{v}$-radial functions on Heisenberg type groups.

Throughout the paper, $\mathbb{N}$ denotes the set of nonnegative integers. Moreover,

$$\mathbb{R}^n_+ = (\mathbb{R}_+)^n = \bigl\{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n \colon \xi_j \geqslant 0,\ j = 1, \dots, n\bigr\}.$$

Finally, if $\mathbf{j} = (j_1, \dots, j_n)$ is a multi-index in $\mathbb{N}^n$, we denote by $|\mathbf{j}|$ its length, i.e., $|\mathbf{j}| = j_1 + \dots + j_n$, and by $\mathbf{j}!$ its factorial, i.e., $\mathbf{j}! = j_1! \cdots j_n!$. As usual, when $x = (x_1, \dots, x_n)$ is in $\mathbb{R}^n$, we denote by $x^{\mathbf{j}}$ the monomial $x_1^{j_1}\cdots x_n^{j_n}$ and by $\partial_x^{\mathbf{j}}$ the differential operator given by the rule $\partial_x^{\mathbf{j}} = \partial_{x_1}^{j_1}\dots\partial_{x_n}^{j_n}$.

2. The Gelfand transform for polyradial functions

We denote by $\mathbb{H}_n$ the Heisenberg group, i.e., the real manifold $\mathbb{C}^n\times\mathbb{R}$ equipped with the group law

$$(z,t)(w,u)=\left(z+w,t+u+\frac{1}{2}\operatorname{Im} w\cdot\bar{z}\right)\quad \forall z,w\in\mathbb{C}^n,\ t,u\in\mathbb{R},$$

where, if $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ are in $\mathbb{C}^n$,

$$w\cdot\bar{z}=\sum_{j=1}^n w_j\overline{z_j}.$$

It is easy to check that Lebesgue measure $dz\,dt$ is a Haar measure on $\mathbb{H}_n$.

The unitary group $\mathrm{U}(n)$, and therefore any of its subgroups, acts on $\mathbb{H}_n$ via

$$k\cdot(z,t)=(kz,t)\quad \forall (z,t)\in\mathbb{H}_n,\ k\in\mathrm{U}(n).$$

We shall be interested in particular in the subgroup $\mathbb{T}^n$ of diagonal matrices of the form $\operatorname{diag}(e^{i\theta_1},\dots,e^{i\theta_n})$, θ_j in $\mathbb{R}$. These actions induce an action on functions f on $\mathbb{H}_n$ by the formula

$$k\cdot f(z,t)=f\big(k^{-1}z,t\big)\quad \forall k\in\mathrm{U}(n),\ (z,t)\in\mathbb{H}_n.$$

We note that a function f on $\mathbb{H}_n$ is $\mathrm{U}(n)$-invariant if it depends only on $|z|$ and t; it is $\mathbb{T}^n$-invariant if it depends only on $|z_1|,\dots,|z_n|$ and t.

We denote by Z_j and $\bar{Z}_j$, where $j=1,\dots,n$, the left-invariant vector fields

$$Z_j=\partial_{z_j}-\frac{i}{4}\bar{z}_j\partial_t,\qquad \bar{Z}_j=\partial_{\bar{z}_j}+\frac{i}{4}z_j\partial_t,\qquad T=\partial_t,$$

satisfying the relation $T=-2i[Z_j,\bar{Z}_j]$. Moreover Z_j and $\bar{Z}_j$ are homogeneous of degree 1 while T is homogeneous of degree 2 with respect to the anisotropic dilations $r\cdot(z,t)=(rz,r^2t)$, where $r>0$ and $(z,t)\in\mathbb{H}_n$. Let $I=(i_1,\dots,i_{2n+1})$ be in $\mathbb{N}^{2n+1}$; we denote by $\mathcal{D}^I$ a differential operator of degree $\deg I=i_1+\dots+i_{2n}+2i_{2n+1}$ of the form

$$\mathcal{D}^I=Z_1^{i_1}\bar{Z}_1^{i_2}\dots Z_n^{i_{2n-1}}\bar{Z}_n^{i_{2n}}T^{i_{2n+1}}.$$

We write $\mathcal{S}(\mathbb{H}_n)$ for the Schwartz space of functions on $\mathbb{H}_n$, i.e., the space of infinitely differentiable functions f on $\mathbb{H}_n$ such that for all partial derivatives $\mathcal{D}^I f$ of f, the function $\mathcal{D}^I f$ is rapidly decreasing. The Schwartz space is equipped with the following family of norms, parametrized by a nonnegative integer p:

$$\|f\|_{(p)}=\sup_{(z,t)\in\mathbb{H}_n}\big\{\big(1+|z|^4+t^2\big)^{p/4}\big|\mathcal{D}^I f(z,t)\big|:\ \deg I\leqslant p\big\}.$$

We denote by $\mathcal{S}_K(\mathbb{H}_n)$ the subspace of K-invariant Schwartz functions.

For the rest of this section, $K=\mathbb{T}^n$. The algebra $L^1_{\mathbb{T}^n}(\mathbb{H}_n)$ of integrable $\mathbb{T}^n$-invariant functions is commutative under convolution [8]. The algebra of left-invariant and $\mathbb{T}^n$-invariant differential operators is generated by $i^{-1}T$ and the partial subLaplacians

$$L_j=-2(Z_j\overline{Z}_j+\overline{Z}_jZ_j),\quad j=1,\dots,n.$$

For β and d in $\mathbb{N}$, let Λ_d^β be the dth Laguerre polynomial of order β, i.e.,

$$\Lambda_d^\beta(u)=\sum_{j=0}^d\binom{d+\beta}{d-j}\frac{(-u)^j}{j!},\quad \forall u\in\mathbb{R}.$$

Moreover, let J_β be the Bessel function of order β, i.e.,

$$J_\beta(u)=\frac{1}{2\pi}\int_0^{2\pi}e^{iu\sin\theta}e^{-i\beta\theta}\,d\theta,\quad \forall u\in\mathbb{R}.$$

The bounded $\mathbb{T}^n$-spherical functions are given by the rules [8]

$$\begin{aligned}\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n}(z,t)&=e^{-\frac14|\lambda||z|^2}\Lambda_{d_1}^0\left(\frac12|\lambda||z_1|^2\right)\cdots\Lambda_{d_n}^0\left(\frac12|\lambda||z_n|^2\right)e^{i\lambda t},\\ \phi_\xi^{\mathbb{T}^n}(z,t)&=J_0\big(\sqrt{\xi_1}|z_1|\big)\cdots J_0\big(\sqrt{\xi_n}|z_n|\big),\quad \forall(z,t)\in\mathbb{H}_n,\end{aligned}$$

where λ is in $\mathbb{R}^*$, $\mathbf{d}=(d_1,\dots,d_n)$ in $\mathbb{N}^n$ and $\xi=(\xi_1,\dots,\xi_n)$ is in $\mathbb{R}^n_+$. To every bounded $\mathbb{T}^n$-spherical function we associate the $(n+1)$-tuple of its eigenvalues with respect to the operators $i^{-1}T,L_1,\dots,L_n$, according to the identities

$$\begin{aligned}&i^{-1}T\big(\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n}\big)=\lambda\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n}, && L_j\big(\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n}\big)=|\lambda|(2d_j+1)\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n},\\ &i^{-1}T\big(\phi_\xi^{\mathbb{T}^n}\big)=0, && L_j\big(\phi_\xi^{\mathbb{T}^n}\big)=\xi_j\phi_\xi^{\mathbb{T}^n}.\end{aligned}\tag{2.1}$$

The Gelfand spectrum $\Sigma_{\mathbb{T}^n}$ is homeomorphic to the Heisenberg brush B_n embedded in $\mathbb{R}^{n+1}$ (see [1]),

$$\Sigma_{\mathbb{T}^n}\simeq B_n=\big\{\big(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\big)\colon\ \lambda\neq0,\ \mathbf{d}\in\mathbb{N}^n\big\}\cup\big\{(0,\xi_1,\dots,\xi_n)\colon\ \xi_j\geqslant0\ \forall j\big\},$$

where $\mathbf{1}=(1,\dots,1)$.

Let f be a function in $L^1_{\mathbb{T}^n}(\mathbb{H}_n)$. We define its Gelfand transform $\mathcal{F}_{\mathbb{T}^n}f$ by the rule

$$\mathcal{F}_{\mathbb{T}^n}f(\phi)=\int_{\mathbb{H}_n}f(z,t)\overline{\phi(z,t)}\,dz\,dt,\quad \forall\phi\in\Sigma_{\mathbb{T}^n}.$$

For the sake of brevity, when $\lambda\neq0$, $\mathbf{d}$ is in $\mathbb{N}^n$ and ξ in $\mathbb{R}^n_+$ we shall more simply write

$$\widetilde{f}\big(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\big)\quad\text{and}\quad\widetilde{f}(0,\xi)$$

instead of $\mathcal{F}_{\mathbb{T}^n}f(\phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n})$ and $\mathcal{F}_{\mathbb{T}^n}f(\phi_\xi^{\mathbb{T}^n})$, respectively.

Note that $\widetilde{f}(0, \xi_1, \dots, \xi_n)$ is the ordinary Euclidean Fourier transform of f at a point $(0, y_1, \dots, y_n)$ in $\mathbb{R} \times \mathbb{C}^n$ with $|y_j|^2 = \xi_j$.

An important tool in our analysis will be a functional calculus result, proved in [15] and based on [6,7,12]. The author is concerned with the kernel k_m of the operator $m(i^{-1}T, L_1, \dots, L_n)$ and proves that the correspondence $m \mapsto k_m$ is continuous from $\mathcal{S}(\mathbb{R}^{n+1})$ to $\mathcal{S}(\mathbb{H}_n)$. Since k_m is $\mathbb{T}^n$-invariant, this result can be rephrased in the language of the Gelfand transform as follows.

Theorem 2.1. *(See [15].) Suppose that m is a Schwartz function on $\mathbb{R}^{n+1}$. Then the restriction to B_n of the function m is the Gelfand transform of a function k_m in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ and the map $m \mapsto k_m$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^{n+1})$ to $\mathcal{S}(\mathbb{H}_n)$.*

The following theorem is contained in [6] as part of the characterization of the image of the Schwartz space under the group Fourier transform. An adaptation of the proof in [6] to $\mathfrak{v}$-radial functions on Heisenberg type groups is in Section 5.

Theorem 2.2. *(See [6].) Let f be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. Then there exist functions f_j, $j \geqslant 1$, in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ such that for any M in $\mathbb{N}$*

$$\widetilde{f}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) = \sum_{j=0}^{M} \frac{\lambda^j}{j!} \widetilde{f_j}\big(0, |\lambda|(2\mathbf{d}+\mathbf{1})\big) + \frac{\lambda^{M+1}}{(M+1)!} \widetilde{f_{M+1}}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big),$$

where $f_0 = f$ and λ is in $\mathbb{R}^*$, $\mathbf{d}$ in $\mathbb{N}^n$. Moreover, for any j, the maps $f \mapsto f_j$ are continuous on $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$.

From now on, given f in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$, we shall denote by f_j the associated functions defined in Theorem 2.2.

Finally, if f is in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$, for any fixed $\mathbf{d}$ in $\mathbb{N}^n$, the map $\lambda \mapsto \widetilde{f}(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1}))$ is smooth when $\lambda \neq 0$. Its derivatives may be estimated using the following classical properties of Laguerre polynomials:

$$\frac{d}{du} \Lambda_{d+1}^{\beta} = \Lambda_d^{\beta+1}, \qquad \big|e^{-u/2} \Lambda_d^{\beta}(u)\big| \leqslant \binom{d+\beta}{d} \quad \forall u \in \mathbb{R},\ \beta, d \in \mathbb{N}.$$

Indeed, from these relations it follows that for every nonnegative integer q

$$\big|\partial_\lambda^q \phi_{\lambda,\mathbf{d}}^{\mathbb{T}^n}(z,t)\big| \leqslant C_q \big(1 + |z|^4 + t^2\big)^{q/2} \big(2|\mathbf{d}| + n\big)^q \quad \forall (z,t) \in \mathbb{H}_n,\ \lambda \neq 0,\ \mathbf{d} \in \mathbb{N}^n.$$

Moreover, let $L = \sum_{j=1}^n L_j$ be the Heisenberg subLaplacian. Then for every nonnegative integer p and $(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1}))$ in B_n (see (2.1)),

$$\big(|\lambda|(2|\mathbf{d}| + n)\big)^p \widetilde{f}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) = \widetilde{L^p f}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big).$$

Therefore, writing

$$\widetilde{f}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) = \big(1 + |\lambda|\big(2|\mathbf{d}| + n\big)\big)^{-p} \big((1+L)^p f\big)^{\sim}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big),$$

one easily obtains for every nonnegative integer p

$$\left|\partial_\lambda^q\left(\widetilde{f}\left(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\right)\right)\right| \leqslant C_{p,q}\frac{(2|\mathbf{d}|+n)^q}{(1+|\lambda|(2|\mathbf{d}|+n))^p}\|f\|_{(2p+2q+2n+3)} \tag{2.2}$$

for every $\lambda \neq 0$ and every $\mathbf{d}$ in $\mathbb{N}^n$.

3. A C^k extension from B_n to $\mathbb{R}^{n+1}$

Let φ_1 be a smooth function on $\mathbb{R}$ such that $\varphi_1(t)=1$ if $|t| \leqslant 1/2$ and $\varphi_1(t)=0$ if $|t| \geqslant 3/4$. We define φ in $C_c^\infty(\mathbb{R}^n)$ by the rule

$$\varphi(x_1,\dots,x_n)=\varphi_1(x_1)\cdots\varphi_1(x_n) \quad \forall (x_1,\dots,x_n)\in\mathbb{R}^n.$$

When h is defined on B_n, we define the function Eh on $\mathbb{R}^{n+1}$ by

$$Eh(\lambda,\xi)=\begin{cases}\sum_{\mathbf{d}\in\mathbb{N}^n} h(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))\varphi(\frac{\xi-|\lambda|(2\mathbf{d}+\mathbf{1})}{|\lambda|}), & \lambda\neq 0,\ \xi\in\mathbb{R}^n,\\ 0, & \lambda=0,\ \xi\in\mathbb{R}^n.\end{cases} \tag{3.1}$$

Note that if $\xi_j<0$ for some j then every term of the series is trivial, and if ξ is in $\mathbb{R}^n_+$ then the series reduces to the sum of at most one term. Therefore Eh is well defined. It coincides with h for $(\lambda,\xi)\in B_n$ and $\lambda\neq 0$ since, using the support properties of the function φ,

$$Eh\left(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\right)=h\left(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\right), \quad \mathbf{d}\in\mathbb{N}^n,\ \lambda\neq 0.$$

Lemma 3.1. *Let k be in $\mathbb{N}$. Suppose that f in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ is such that*

$$\widetilde{f}_j(0,\xi)=0 \quad \forall (0,\xi)\in B_n,\ \forall j=0,\dots,2k.$$

Then the function $E\widetilde{f}$, defined as in (3.1)*, is in $C^k(\mathbb{R}^{n+1})$ and*

(1) $E\widetilde{f}(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))=\widetilde{f}(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))$ $\forall\lambda\in\mathbb{R}\setminus\{0\}$, $\mathbf{d}\in\mathbb{N}^n$;
(2) $\partial_\lambda^s(E\widetilde{f})(0,\xi)=0$ $\forall\xi\in\mathbb{R}^n$, $0\leqslant s\leqslant k$;
(3) *for every $p\geqslant 0$ there exist a constant $C_{k,p}$ and a Schwartz norm $\|\cdot\|_{(p')}$ such that*

$$\left(1+|(\lambda,\xi)|\right)^p\left|\partial_\lambda^s\partial_\xi^{\mathbf{j}}(E\widetilde{f})(\lambda,\xi)\right|\leqslant C_{k,p}\|f\|_{(p')} \quad \forall(\lambda,\xi)\in\mathbb{R}^{n+1},\ s+|\mathbf{j}|\leqslant k.$$

Proof. As we have already remarked, the series (3.1) consists locally of at most one single term and (1) holds. For any fixed $\mathbf{d}$ in $\mathbb{N}^n$ the map $\lambda\mapsto\widetilde{f}(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))$ is smooth when $\lambda\neq 0$. Therefore the function $E\widetilde{f}$ is smooth when $\lambda\neq 0$.

We now show that $E\widetilde{f}$ is of class $C^k(\mathbb{R}^{n+1})$ and (2) holds. Note that, by Theorem 2.2 and by hypothesis, $\widetilde{f}(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))=\frac{\lambda^{2k+1}}{(2k+1)!}\widetilde{f_{2k+1}}(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1}))$, hence

$$E\widetilde{f}(\lambda,\xi)=\sum_{\mathbf{d}\in\mathbb{N}^n}\frac{\lambda^{2k+1}}{(2k+1)!}\widetilde{f_{2k+1}}\left(\lambda,|\lambda|(2\mathbf{d}+\mathbf{1})\right)\varphi\left(\frac{\xi-|\lambda|(2\mathbf{d}+\mathbf{1})}{|\lambda|}\right).$$

We apply the differential operator $\partial_\lambda^s \partial_\xi^{\mathbf{j}}$, with $s + |\mathbf{j}| \leqslant k$, to each term in the sum. In the case where $0 < |\lambda| < 1$ and ξ is in $\mathbb{R}^n$, it is easy to check that by the estimate (2.2)

$$\left| \sum_{\mathbf{d}\in\mathbb{N}^n} \partial_\lambda^s \partial_\xi^{\mathbf{j}} \left(\frac{\lambda^{2k+1}}{(2k+1)!} \widetilde{f_{2k+1}}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \varphi\left(\frac{\xi - |\lambda|(2\mathbf{d}+\mathbf{1})}{|\lambda|} \right) \right) \right|$$
$$\leqslant C_{\mathbf{j},s} |\lambda|^{2k+1-2s-|\mathbf{j}|} \big(1+|\xi|\big)^s \sum_{\ell=0}^{s} \| f_{2k+1} \|_{(2\ell+2n+3)}.$$

If $s + |\mathbf{j}| \leqslant k$, then $2k+1-2s-|\mathbf{j}| > 0$ and when (λ, ξ) tends to $(0, \xi_0)$, the above term converges to 0 for every $\xi_0 \in \mathbb{R}^n$. Therefore $E\widetilde{f}$ is in $C^k(\mathbb{R}^{n+1})$ and

$$\partial_\lambda^s E\widetilde{f}(0,\xi) = 0 \quad \forall \xi \in \mathbb{R}^n,\ 0 \leqslant s \leqslant k.$$

Now we prove (3). Since the function $E\widetilde{f}$ is supported in the set

$$\big\{(\lambda,\xi) \in \mathbb{R}^{n+1}\colon\ \xi_j \geqslant |\lambda|/4\ \forall j\big\},$$

it suffices to show that it is rapidly decreasing in the ξ-variables, together with all its derivatives up to order k.

Let $s + |\mathbf{j}| \leqslant k$ and p in $\mathbb{N}$. For any fixed (λ, ξ) in $\mathbb{R} \times \mathbb{R}_+^n$ there exists at most one $\mathbf{d}$ in $\mathbb{N}^n$ such that $\frac{|\xi_i - |\lambda|(2d_i+1)|}{|\lambda|} \leqslant \frac{3}{4}$ for every $i = 1, \dots, n$, and in this case

$$\frac{1}{4} \leqslant \frac{\xi_i}{|\lambda|(2d_i+1)} \leqslant \frac{7}{4} \quad \forall i = 1, \dots, n.$$

Note that $(1+|\xi|)^p |\partial_\lambda^s \partial_\xi^{\mathbf{j}} (E\widetilde{f})(\lambda,\xi)|$ is controlled by a finite linear combination of terms of the form

$$|\lambda|^q \big(1+|\xi|\big)^{p+s} \big| \partial_\lambda^r \widetilde{f_{2k+1}}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \big|,$$

where $q > r \geqslant 0$. Thus (3) follows from formula (2.2) and Theorem 2.2. $\square$

We remove now the assumptions on the vanishing of the functions $\widetilde{f}_j(0,\cdot)$ in Lemma 3.1.

Arguing as in the proof of the Whitney–Stein Extension Theorem [13, p. 181], one can easily show that if f is any function in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$, then the function $\xi \mapsto \widetilde{f}(0,\xi)$ can be extended to a Schwartz function on all of $\mathbb{R}^n$ in a linear and continuous way. We shall keep the same notation $\widetilde{f}(0,\cdot)$ for the so-extended function.

Moreover, let Ψ be a smooth function on $\mathbb{R}^{n+1}$ with bounded derivatives, such that $\Psi(\lambda,\xi) = 1$ if (λ,ξ) is in the convex hull $\mathrm{co}(B_n)$ of the Heisenberg brush B_n and $\Psi(\lambda,\xi) = 0$ if $\mathrm{dist}((\lambda,\xi), \mathrm{co}(B_n)) > 1$.

Proposition 3.2. *Let f be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. Then, for every integer $k \geqslant 0$, the function $\widetilde{f}$ can be extended to a function Φ_k in $C^k(\mathbb{R}^{n+1})$ such that*:

(1) $\Phi_k(\lambda,\xi) = \widetilde{f}(\lambda,\xi)\ \forall (\lambda,\xi) \in B_n$;

(2) $\partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_k(0,\xi) = \partial_\xi^{\mathbf{j}}(\Psi \widetilde{f}_s)(0,\xi)$ $\forall \xi \in \mathbb{R}^n$, $s + |\mathbf{j}| \leqslant k$;
(3) $\partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_k(0,\xi) = \partial_\xi^{\mathbf{j}} \widetilde{f}_s(0,\xi)$ $\forall \xi \in \mathbb{R}_+^n$, $s + |\mathbf{j}| \leqslant k$;
(4) *for every $p \geqslant 0$ there exist a constant $C_{k,p}$ and a Schwartz norm $\|\cdot\|_{(p')}$ such that*

$$\left(1 + \left|(\lambda,\xi)\right|\right)^p \left|\partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_k(\lambda,\xi)\right| \leqslant C_{k,p} \|f\|_{(p')} \quad \forall (\lambda,\xi) \in \mathbb{R}^{n+1},\ s + |\mathbf{j}| \leqslant k.$$

Proof. Fix k in $\mathbb{N}$. Let g be the function in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ defined by

$$g = \frac{1}{(2k+1)!} \left(i^{-1} T\right)^{2k+1} f_{2k+1}.$$

Then for every $\lambda \neq 0$ and $\mathbf{d}$ in $\mathbb{N}^n$,

$$\widetilde{g}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) = \frac{\lambda^{2k+1}}{(2k+1)!} \widetilde{f}_{2k+1}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \quad \forall \big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \in B_n,$$

so $\widetilde{g}_j(0,\xi) = 0$ for every ξ in $\mathbb{R}_+^n$ and every $j = 0, \ldots, 2k$.

Define

$$\Phi_k(\lambda,\xi) = \left(\sum_{s=0}^{2k} \frac{\lambda^s}{s!} \widetilde{f}_s(0,\xi) + E\widetilde{g}(\lambda,\xi) \right) \Psi(\lambda,\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathbb{R}^n.$$

Then, by Lemma 3.1, the function Φ_k is of class $C^k(\mathbb{R}^{n+1})$ and for every $\lambda \neq 0$ and $\mathbf{d}$ in $\mathbb{N}^n$

$$\begin{aligned}
\Phi_k\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) &= \sum_{j=0}^{2k} \frac{\lambda^j}{j!} \widetilde{f}_j\big(0, |\lambda|(2\mathbf{d}+\mathbf{1})\big) + E\widetilde{g}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \\
&= \sum_{j=0}^{2k} \frac{\lambda^j}{j!} \widetilde{f}_j\big(0, |\lambda|(2\mathbf{d}+\mathbf{1})\big) + \frac{\lambda^{2k+1}}{(2k+1)!} \widetilde{f_{2k+1}}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big) \\
&= \widetilde{f}\big(\lambda, |\lambda|(2\mathbf{d}+\mathbf{1})\big).
\end{aligned}$$

Moreover, when $\mathbf{j}$ is in $\mathbb{N}^n$, s in $\mathbb{N}$, $s + |\mathbf{j}| \leqslant k$, and $\xi \in \mathbb{R}^n$,

$$\partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_k(0,\xi) = \partial_\xi^{\mathbf{j}}(\widetilde{f}_s \Psi)(0,\xi) + \partial_\xi^{\mathbf{j}}\big(\partial_\lambda^s E\widetilde{g}\Psi\big)(0,\xi) = \partial_\xi^{\mathbf{j}}(\widetilde{f}_s \Psi)(0,\xi).$$

From this formula (3) follows easily. Finally, (4) follows from Lemma 3.1, Theorem 2.2, and the support properties of the function Ψ. $\square$

4. Approximation by smooth rapidly decreasing functions

In this section we adapt some arguments that can be found in [11, Chapter 1]. We give details in order to obtain explicit bounds in terms of the various Schwartz seminorms.

The next lemma deals with functions defined on $\mathbb{R}^m \times \mathbb{R}^n$. It will be used for polyradial functions on $\mathbb{H}_n$ in the case where $m = 1$ and for $\mathfrak{v}$-radial functions on H-type groups in the case where $m \geqslant 1$ and $n = 1$.

The coordinates in $\mathbb{R}^m \times \mathbb{R}^n$ will be denoted by (λ, ξ), with λ in $\mathbb{R}^m$ and ξ in $\mathbb{R}^n$. For $\mathbf{j} = (\mathbf{j}', \mathbf{j}'')$ a multi-index in $\mathbb{N}^{m+n}$, we denote by $D^{\mathbf{j}}$ the differential operator $\partial_\lambda^{\mathbf{j}'} \partial_\xi^{\mathbf{j}''}$.

Lemma 4.1. *Let $k \geqslant 1$ and suppose that h is a C^k function on $\mathbb{R}^m \times \mathbb{R}^n$ such that*:

(1) $D^{\mathbf{j}} h(0, \xi) = 0$, $0 \leqslant |\mathbf{j}| \leqslant k$ $\forall \xi \in \mathbb{R}^n$;

(2) *for every integer $p \geqslant 0$ there exists a constant $\alpha_{k,p}(h)$ such that*

$$\sup_{|\mathbf{j}| \leqslant k} \left\| \left(1 + |\cdot|\right)^p D^{\mathbf{j}} h \right\|_\infty \leqslant \alpha_{k,p}(h).$$

Then for every $\varepsilon > 0$ and for every integer $M \geqslant 0$ there exists a function $h_{\varepsilon,M}$ in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ such that:

(1′) $D^{\mathbf{j}} h_{\varepsilon,M}(0, \xi) = 0$ $\forall \mathbf{j} \in \mathbb{N}^{m+n}$, $\forall \xi \in \mathbb{R}^n$;

(2′) $\sup_{|\mathbf{j}| \leqslant k-1} \|(1 + |\cdot|)^M D^{\mathbf{j}}(h - h_{\varepsilon,M})\|_\infty < \varepsilon$;

(3′) *for every p in $\mathbb{N}$ there exists a constant C_{kMp} such that*

$$\sup_{|\mathbf{j}| \leqslant p} \left\| \left(1 + |\cdot|\right)^p D^{\mathbf{j}}(h_{\varepsilon,M}) \right\|_\infty \leqslant C_{kMp} \left\| \left(1 + |\cdot|\right)^p h \right\|_\infty \max\left\{1, C_{kMp} \alpha_{k,M}^p(h) \varepsilon^{-p}\right\}.$$

Proof. Suppose that δ is a positive real number to be chosen afterwards. We fix two smooth functions χ and η with the following properties.

The function χ depends only on λ, $\chi(\lambda) = 0$ if $|\lambda| < 1$ and $\chi(\lambda) = 1$ if $|\lambda| \geqslant 2$. Then $\chi^\delta(\lambda) = \chi(\delta^{-1}\lambda)$ is in $C^\infty(\mathbb{R}^m)$, $\chi^\delta(\lambda) = 0$ if $|\lambda| < \delta$ and $\chi^\delta(\lambda) = 1$ if $|\lambda| \geqslant 2\delta$. Moreover for every $\mathbf{q} = (q_1, \ldots, q_m)$ in $\mathbb{N}^m$,

$$\left|\partial_\lambda^{\mathbf{q}} \chi^\delta\right| \leqslant \left\|\partial_\lambda^{\mathbf{q}} \chi\right\|_\infty \delta^{-|\mathbf{q}|}. \tag{4.1}$$

The function η is a smooth nonnegative function on $\mathbb{R}^m \times \mathbb{R}^n$ with compact support contained in the unit ball and such that $\|\eta\|_1 = 1$. For every $r > 0$, let η_r be the function defined by the rule

$$\eta_r(\lambda, \xi) = r^{-m-n} \eta(\lambda/r, \xi/r) \quad \forall (\lambda, \xi) \in \mathbb{R}^m \times \mathbb{R}^n.$$

Then $\eta_{\delta/2}$ is a smooth nonnegative function on $\mathbb{R}^m \times \mathbb{R}^n$ with compact support contained in the ball of radius $\delta/2$ and such that $\|\eta_{\delta/2}\|_1 = 1$.

Fix $\varepsilon > 0$ and a nonnegative integer M and define

$$h_{\varepsilon,M} = \left(h\chi^\delta\right) * \eta_{\delta/2}.$$

We will choose δ, depending on ε and M, small enough so that $h_{\varepsilon,M}$ satisfies the required conditions.

First note that the function $h_{\varepsilon,M}$ is smooth, because $\eta_{\delta/2}$ is smooth. Moreover, for every $\mathbf{j}$ in $\mathbb{N}^{m+n}$, the function $D^{\mathbf{j}} h_{\varepsilon,M} = (h\chi^\delta) * D^{\mathbf{j}}\eta_{\delta/2}$ is the convolution of two rapidly decreasing functions. Therefore $h_{\varepsilon,M}$ is a Schwartz function on $\mathbb{R}^m \times \mathbb{R}^n$.

The function $h_{\varepsilon,M}$ is identically 0 in $\{(\lambda,\xi) \in \mathbb{R}^m \times \mathbb{R}^n \colon |\lambda| < \delta/2\}$. Therefore

$$D^{\mathbf{j}} h_{\varepsilon,M}(0,\xi) = 0 \quad \forall \mathbf{j} \in \mathbb{N}^{m+n},\ \xi \in \mathbb{R}^n.$$

Hence condition (1′) holds for any choice of δ.

We now check condition (2′). For convenience, we shall suppose that $\delta < 1$. Let $\mathbf{j}$ be in $\mathbb{N}^{m+n}$ such that $|\mathbf{j}| \leqslant k-1$. Then

$$\begin{aligned} D^{\mathbf{j}}(h_{\varepsilon,M} - h) &= D^{\mathbf{j}}\big((h\chi^\delta) * \eta_{\delta/2}\big) - D^{\mathbf{j}} h \\ &= \big(D^{\mathbf{j}} h\big) * \eta_{\delta/2} - D^{\mathbf{j}} h - \big(D^{\mathbf{j}}\big(h\big(1-\chi^\delta\big)\big)\big) * \eta_{\delta/2}. \end{aligned}$$

We estimate the terms $(D^{\mathbf{j}} h) * \eta_{\delta/2} - D^{\mathbf{j}} h$ and $(D^{\mathbf{j}}(h(1-\chi^\delta))) * \eta_{\delta/2}$ separately.

Using the properties of the approximate identity η and the Mean Value Theorem, it is easy to show that for every (λ,ξ) in $\mathbb{R}^m \times \mathbb{R}^n$ and for any multi-index $\mathbf{j}$ in $\mathbb{N}^{m+n}$ such that $|\mathbf{j}| \leqslant k-1$,

$$\big(1+\big|(\lambda,\xi)\big|\big)^M \big|\big[\big(D^{\mathbf{j}} h\big) * \eta_{\delta/2} - D^{\mathbf{j}} h\big](\lambda,\xi)\big| \leqslant 2^{M-1}\alpha_{k,M}(h)\delta.$$

This estimates the first term.

To treat the second term, we observe that, by hypothesis (1), we may estimate the derivatives of the function h using Taylor's formula in the first set of variables. Indeed, when $|\mathbf{p}| \leqslant k$, then for every (λ,ξ) in $\mathbb{R}^m \times \mathbb{R}^n$ and for a suitable ϑ in $(0,1)$,

$$\big|D^{\mathbf{p}} h(\lambda,\xi)\big| = \left|\sum_{|\mathbf{q}|=k-|\mathbf{p}|} \partial_\lambda^{\mathbf{q}} D^{\mathbf{p}} h(\vartheta\lambda,\xi)\frac{\lambda^{\mathbf{q}}}{\mathbf{q}!}\right| \leqslant C_k|\lambda|^{k-|\mathbf{p}|}\big(1+|\xi|\big)^{-M}\alpha_{k,M}(h).$$

Moreover, since χ does not depend on ξ, we can use the Leibniz rule, formula (4.1), and the previous estimate, to obtain that for every (λ,ξ) in $\mathbb{R}^m \times \mathbb{R}^n$,

$$\big|D^{\mathbf{j}}\big(h\big(1-\chi^\delta\big)\big)(\lambda,\xi)\big| \leqslant C_k|\lambda|^{k-|\mathbf{j}|}\big(1+|\xi|\big)^{-M}\alpha_{k,M}(h).$$

Finally, note that if $|t| \leqslant 2\delta$ and $|(\lambda-t,\xi-s)| \leqslant \delta/2$, then $1+|(\lambda,\xi)| \leqslant 4(1+|s|)$, because $\delta < 1$. Since $|\mathbf{j}| \leqslant k-1$, putting together all these estimates, we obtain that, for every (λ,ξ) in $\mathbb{R}^m \times \mathbb{R}^n$,

$$\begin{aligned} &\big(1+\big|(\lambda,\xi)\big|\big)^M \big|\big[\big(D^{\mathbf{j}}\big(h\big(1-\chi^\delta\big)\big)\big) * \eta_{\delta/2}\big](\lambda,\xi)\big| \\ &\quad = \big(1+\big|(\lambda,\xi)\big|\big)^M \left|\int_{|t|\leqslant 2\delta} \big(D^{\mathbf{j}}\big(h\big(1-\chi^\delta\big)\big)\big)(t,s)\eta_{\delta/2}(\lambda-t,\xi-s)\,dt\,ds\right| \\ &\quad \leqslant C_{k,M}\alpha_{k,M}(h) \int_{|t|\leqslant 2\delta} |t|^{k-|\mathbf{j}|}\eta_{\delta/2}(\lambda-t,\xi-s)\,dt\,ds \end{aligned}$$

$$\begin{aligned}&\leqslant C_{k,M}\alpha_{k,M}(h)\delta \int\limits_{\mathbb{R}^m\times\mathbb{R}^n} \eta_{\delta/2}(\lambda-t,\xi-s)\,dt\,ds\\&= C_{k,M}\alpha_{k,M}(h)\delta.\end{aligned}$$

This estimates the second term.

We conclude that there exists a constant $C_{k,M}$ such that, if

$$\delta < \min\{1, \varepsilon C_{k,M}\alpha_{k,M}^{-1}(h)\},$$

then, for all multi-index $\mathbf{j}$ in $\mathbb{N}^{m+n}$ with $|\mathbf{j}| \leqslant k-1$,

$$\left\|\big(1+|\cdot|\big)^M D^{\mathbf{j}}(h_{\varepsilon,M}-h)\right\|_\infty < \varepsilon,$$

i.e., condition (2′) is satisfied.

Finally we show that (3′) holds for this choice of δ. Indeed, for any p in $\mathbb{N}$, any multi-index $\mathbf{j}$ in $\mathbb{N}^{m+n}$ with $|\mathbf{j}| \leqslant p$ and for every (λ,ξ) in $\mathbb{R}^m\times\mathbb{R}^n$,

$$\begin{aligned}\big(1+\big|(\lambda,\xi)\big|\big)^p\big|D^{\mathbf{j}}h_{\varepsilon,M}(\lambda,\xi)\big| &= \big(1+\big|(\lambda,\xi)\big|\big)^p\big|\big(h\chi^\delta\big)*D^{\mathbf{j}}\eta_{\delta/2}(\lambda,\xi)\big|\\&= \big(1+\big|(\lambda,\xi)\big|\big)^p\left|\int\limits_{|t|\geqslant\delta}\big(h\chi^\delta\big)(t,s)D^{\mathbf{j}}\eta_{\delta/2}(\lambda-t,\xi-s)\,dt\,ds\right|\\&\leqslant \int\limits_{|t|\geqslant\delta} 2^p\big(1+\big|(t,s)\big|\big)^p\big|\big(h\chi^\delta\big)(t,s)\big|\big|D^{\mathbf{j}}\eta_{\delta/2}(\lambda-t,\xi-s)\big|\,dt\,ds\\&\leqslant 2^p\big\|\big(1+|\cdot|\big)^p h\big\|_\infty \int\limits_{\mathbb{R}^m\times\mathbb{R}^n}(\delta/2)^{-|\mathbf{j}|}\big|\big(D^{\mathbf{j}}\eta\big)_{\delta/2}(t,s)\big|\,dt\,ds\\&\leqslant 4^p\big\|\big(1+|\cdot|\big)^p h\big\|_\infty\big\|D^{\mathbf{j}}\eta\big\|_1\delta^{-p},\end{aligned}$$

where, as before, for any $r>0$ we put $(D^{\mathbf{j}}\eta)_r(t,s) = r^{-m-n}D^{\mathbf{j}}\eta(t/r,s/r)$ for all (t,s) in $\mathbb{R}^m\times\mathbb{R}^n$. $\square$

In the next proposition we return to polyradial functions on the Heisenberg group $\mathbb{H}_n$. For any fixed nonnegative integer p, we apply Lemma 4.1 to build a Schwartz function on $\mathbb{R}^{n+1}$ with preassigned values on the ξ-hyperplane, together with all its derivatives, and whose Schwartz seminorms can be controlled up to order p. We remark that if p is a nonnegative integer, $\|\cdot\|_{(p)}$ is the p-order Schwartz norm of functions on $\mathbb{H}_n$; we do not use a special notation for the Schwartz norm of functions on $\mathbb{R}^{n+1}$.

Proposition 4.2. *Suppose that f is in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. Then for any p in $\mathbb{N}$ there exist a Schwartz function H on $\mathbb{R}^{n+1}$ and q in $\mathbb{N}$, both depending on p, such that*

$$\partial_\lambda^s\partial_\xi^{\mathbf{j}}H(0,\xi) = \partial_\xi^{\mathbf{j}}\widetilde{f}_s(0,\xi) \quad \forall\xi\in\mathbb{R}^n_+,\ (s,\mathbf{j})\in\mathbb{N}^{n+1}$$

and

$$\sup_{s+|\mathbf{j}|\leqslant p} \left\| \left(1+|\cdot|\right)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \leqslant C_p \|f\|_{(q)}.$$

Proof. Let f be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. For any k in $\mathbb{N}$, let Φ_k be the C^k-extension of $\widetilde{f}$ constructed in Proposition 3.2. Then the function $\Phi_{k+1} - \Phi_k$ is in $C^k(\mathbb{R}^{n+1})$ and

$$\partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k)(0,\xi) = 0 \quad \forall \xi \in \mathbb{R}^n,\ 0 \leqslant s + |\mathbf{j}| \leqslant k.$$

Moreover for every r in $\mathbb{N}$ there exists q_r in $\mathbb{N}$ such that

$$\sup_{0\leqslant s+|\mathbf{j}|\leqslant k\leqslant r} \left\| \left(1+|\cdot|\right)^r \partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_k \right\|_\infty \leqslant C_r \|f\|_{(q_r)},$$

$$\alpha_r = \sup_{s+|\mathbf{j}|\leqslant k\leqslant r} \left\| \left(1+|\cdot|\right)^r \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k) \right\|_\infty \leqslant C_r \|f\|_{(q_r)}. \tag{4.2}$$

Fix p in $\mathbb{N}$. For $k \geqslant 1$ we apply Lemma 4.1 to the function $\Phi_{k+1} - \Phi_k$ with $\varepsilon = 2^{-k}\|f\|_{(q_p)}$ and $M = k$ and we conclude that there exists a Schwartz function H_k on $\mathbb{R}^{n+1}$ such that

$$\partial_\lambda^s \partial_\xi^{\mathbf{j}} H_k(0,\xi) = 0 \quad \forall s \in \mathbb{N},\ \forall \mathbf{j} \in \mathbb{N}^n,\ \forall \xi \in \mathbb{R}^n$$

and

$$\sup_{s+|\mathbf{j}|\leqslant k-1} \left\| \left(1+|\cdot|\right)^k \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k - H_k) \right\|_\infty < 2^{-k}\|f\|_{(q_p)}. \tag{4.3}$$

Then we can define a function H on $\mathbb{R}^{n+1}$ by the rule

$$H = \Phi_1 + \sum_{k=1}^{+\infty} (\Phi_{k+1} - \Phi_k - H_k).$$

Note that for every $r \geqslant 1$,

$$H = \Phi_r - \sum_{k=1}^{r-1} H_k + \sum_{k=r}^{+\infty} (\Phi_{k+1} - \Phi_k - H_k).$$

Hence by formula (4.3), for every $r \geqslant 1$ and any multi-index $(s,\mathbf{j})$ in $\mathbb{N}^{n+1}$ of length at most $r-1$, the series $\sum_{k=r}^{+\infty} \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k - H_k)(\lambda,\xi)$ is uniformly convergent on $\mathbb{R}^{n+1}$. This means that H is a smooth function on $\mathbb{R}^{n+1}$ and, for every r in $\mathbb{N}$,

$$\partial_\lambda^s \partial_\xi^{\mathbf{j}} H(0,\xi) = \partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_r(0,\xi) = \partial_\xi^{\mathbf{j}} (\widetilde{f}_s \Psi)(0,\xi) \quad \forall \xi \in \mathbb{R}^n,\ (s,\mathbf{j}) \in \mathbb{N}^{n+1},\ s + |\mathbf{j}| \leqslant r.$$

Since r is arbitrary, we conclude that $\partial_\lambda^s \partial_\xi^{\mathbf{j}} H(0,\xi) = \partial_\xi^{\mathbf{j}} (\widetilde{f}_s \Psi)(0,\xi)$ for every $(s,\mathbf{j})$ in $\mathbb{N}^{n+1}$ and ξ in $\mathbb{R}^n$, while if ξ is in $\mathbb{R}^n_+$, then $\partial_\lambda^s \partial_\xi^{\mathbf{j}} H(0,\xi) = \partial_\xi^{\mathbf{j}} \widetilde{f}_s(0,\xi)$.

We now check that H and all its derivatives are rapidly decreasing. Fix ℓ in $\mathbb{N}$ and let $(s,\mathbf{j})$ be in $\mathbb{N}^{n+1}$ such that $s + |\mathbf{j}| \leqslant \ell$. Then

$$\begin{aligned}
&\big(1+\big|(\lambda,\xi)\big|\big)^\ell \big|\big(\partial_\lambda^s \partial_\xi^{\mathbf{j}} H\big)(\lambda,\xi)\big| \\
&\quad = \big(1+\big|(\lambda,\xi)\big|\big)^\ell \left| \left[\partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_{\ell+1} - \sum_{k=1}^{\ell} \partial_\lambda^s \partial_\xi^{\mathbf{j}} H_k + \sum_{k=\ell+1}^{+\infty} \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k - H_k) \right](\lambda,\xi) \right|.
\end{aligned}$$

We estimate separately the three terms above. By (4.2)

$$\big\|\big(1+|\cdot|\big)^\ell \partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi_{\ell+1}\big\|_\infty \leqslant C_\ell \|f\|_{(q_\ell)}$$

and by formula (4.3)

$$\begin{aligned}
&\sum_{k=\ell+1}^{+\infty} \big\|\big(1+|\cdot|\big)^\ell \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k - H_k)\big\|_\infty \\
&\quad \leqslant \sum_{k=\ell+1}^{+\infty} \big\|\big(1+|\cdot|\big)^k \partial_\lambda^s \partial_\xi^{\mathbf{j}} (\Phi_{k+1} - \Phi_k - H_k)\big\|_\infty \\
&\quad \leqslant \sum_{k=\ell+1}^{+\infty} 2^{-k} \|f\|_{(q_p)} = 2^{-\ell-1} \|f\|_{(q_p)}.
\end{aligned}$$

Finally, by (3′) in Lemma 4.1 and by (4.2), when $k = 1, \ldots, \ell-1$,

$$\begin{aligned}
\big\|\big(1+|\cdot|\big)^\ell \partial_\lambda^s \partial_\xi^{\mathbf{j}} H_k\big\|_\infty &\leqslant C_\ell \big\|\big(1+|\cdot|\big)^\ell (\Phi_{k+1} - \Phi_k)\big\|_\infty \max\big\{1, C_\ell \|f\|_{(q_p)}^{-\ell} \alpha_\ell^\ell\big\} \\
&\leqslant C_\ell \|f\|_{(q_\ell)} \max\big\{1, C_p \|f\|_{(q_p)}^{-\ell} \|f\|_{(q_\ell)}^\ell\big\}.
\end{aligned}$$

Therefore H is a Schwartz function and taking $\ell = p$ we prove the required control on its p-order Schwartz norm. $\square$

In the next theorem we conclude the proof of Theorem 1.1 in the polyradial case.

Theorem 4.3. *Let f be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. Then for every p in $\mathbb{N}$ there exist q in $\mathbb{N}$ and Φ in $\mathcal{S}(\mathbb{R}^{n+1})$ such that*

$$\Phi(\lambda,\xi) = \widetilde{f}(\lambda,\xi) \quad \forall (\lambda,\xi) \in B_n$$

and

$$\sup_{|\mathbf{j}| \leqslant p} \big\|\big(1+|\cdot|\big)^p D^{\mathbf{j}} \Phi\big\|_\infty \leqslant C_p \|f\|_{(q)}.$$

Proof. Fix p in $\mathbb{N}$. By Lemma 3.1, there exists $p' \geqslant p$ such that

$$\sup_{s+|\mathbf{j}|\leqslant p} \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} E\widetilde{g} \right\|_\infty \leqslant C_p \|g\|_{(p')}, \tag{4.4}$$

for every g in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ such that $\widetilde{g}_j(0,\cdot) = 0$ for all $j \geqslant 0$.

Moreover, by Theorem 2.1, given p' there exists $p'' \geqslant p'$ such that

$$\|h\|_{(p')} \leqslant C_{p'} \sup_{s+|\mathbf{j}|\leqslant p''} \left\| (1+|\cdot|)^{p''} \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty, \tag{4.5}$$

for every H in $\mathcal{S}(\mathbb{R}^{n+1})$ and h in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ such that $\widetilde{h} = H_{|B_n}$.

Let f be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$. We want to find Φ in $\mathcal{S}(\mathbb{R}^{n+1})$ with a control on its p-Schwartz norm such that $\widetilde{f} = \Phi_{|B_n}$.

By Proposition 4.2, there exist a function H in $\mathcal{S}(\mathbb{R}^{n+1})$ and an integer $q \geqslant p''$ such that

$$\begin{gathered} \partial_\lambda^s \partial_\xi^{\mathbf{j}} H(0,\xi) = \partial_\xi^{\mathbf{j}} \widetilde{f}_s(0,\xi) \quad \forall \xi \in \mathbb{R}_+^n,\ (s,\mathbf{j}) \in \mathbb{N}\times\mathbb{N}^n, \\ \sup_{s+|\mathbf{j}|\leqslant p''} \left\| (1+|\cdot|)^{p''} \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \leqslant C_p \|f\|_{(q)}. \end{gathered} \tag{4.6}$$

Let h be in $\mathcal{S}_{\mathbb{T}^n}(\mathbb{H}_n)$ such that $\widetilde{h} = H_{|B_n}$, which exists by Theorem 2.1, and define $g = f - h$. Then it is easy to verify that $\widetilde{g}_j(0,\cdot) = 0$ for all $j \geqslant 0$.

Finally, define $\Phi = E\widetilde{g} + H$. Then Φ is in $\mathcal{S}(\mathbb{R}^{n+1})$ and, when restricted to the Heisenberg brush B_n, coincides with $\widetilde{f}$. Moreover, by (4.4)–(4.6), for every multi-index $\mathbf{j}$ in $\mathbb{N}^n$ and every nonnegative integer s with $s + |\mathbf{j}| \leqslant p$,

$$\begin{aligned} \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} \Phi \right\|_\infty &\leqslant \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} E\widetilde{g} \right\|_\infty + \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \\ &\leqslant C_p \|g\|_{(p')} + \sup_{s+|\mathbf{j}|\leqslant p} \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \\ &\leqslant C_p \big(\|f\|_{(p')} + \|h\|_{(p')} \big) + \sup_{s+|\mathbf{j}|\leqslant p} \left\| (1+|\cdot|)^p \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \\ &\leqslant C_p \Big(\|f\|_{(p')} + \sup_{s+|\mathbf{j}|\leqslant p''} \left\| (1+|\cdot|)^{p''} \partial_\lambda^s \partial_\xi^{\mathbf{j}} H \right\|_\infty \Big) \\ &\leqslant C_p \|f\|_{(q)}, \end{aligned}$$

as required. □

Remark. Note that in the previous proof, the function Φ depends on p, because the function H, coming from Proposition 4.2, depends on p. We do not know whether Φ can be chosen independently of p.

5. Heisenberg type groups

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle,\rangle_{\mathfrak{n}}$. Write $\mathfrak{n}$ as an orthogonal sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$.

For each t in $\mathfrak{z}$, define the map $J_t : \mathfrak{v} \to \mathfrak{v}$ by the formula

$$\langle J_t x, y\rangle_{\mathfrak{n}} = \big\langle [x, y], t\big\rangle_{\mathfrak{n}} \quad \forall x, y \in \mathfrak{v}.$$

According to A. Kaplan [9], the Lie algebra $\mathfrak{n}$ is said to be H-type if, for every t in $\mathfrak{z}$,

$$J_t^2 = -\langle t, t\rangle_{\mathfrak{n}} I_{\mathfrak{v}},$$

where $I_{\mathfrak{v}}$ is the identity on $\mathfrak{v}$. A connected and simply connected Lie group N whose Lie algebra is an H-type algebra is said to be a Heisenberg type group, or H-type group for short.

Since $\mathfrak{n}$ is a nilpotent Lie algebra, the exponential map is surjective. We can then parametrise the elements of $N = \exp \mathfrak{n}$ by (x, t), for x in $\mathfrak{v}$ and t in $\mathfrak{z}$. By the Baker–Campbell–Hausdorff formula it follows that the product law in N is

$$(x, t)(x', t') = \left(x + x', t + t' + \frac{1}{2}[x, x']\right) \quad \forall x, x' \in \mathfrak{v},\ \forall t, t' \in \mathfrak{z}.$$

We denote by dx and dt the Lebesgue measures on $\mathfrak{v}$ and on $\mathfrak{z}$ respectively; it is easy to check that $dn = dx\,dt$ is a Haar measure on N.

Note that for every unit vector t in $\mathfrak{z}$, the map J_t defines a complex structure on $\mathfrak{v}$; therefore $\mathfrak{v}$ has even dimension, $2n$ say. We denote the dimension of the center $\mathfrak{z}$ by m.

Let $\{X_j\}_{j=1}^{2n}$ and $\{T_j\}_{j=1}^{m}$ be the left-invariant vector fields corresponding to chosen orthonormal bases of $\mathfrak{v}$ and of $\mathfrak{z}$, respectively. According to this choice of orthonormal bases, we shall identify $\mathfrak{z}$ and its dual $\mathfrak{z}^*$ with $\mathbb{R}^m$ and, for t in $\mathfrak{z}$ and λ in $\mathfrak{z}^*$, we will write $t = (t_1, \dots, t_m)$ and $\lambda = (\lambda_1, \dots, \lambda_m)$.

We say that a function f on N is $\mathfrak{v}$-radial if there exists a function f_0 defined on $\mathbb{R} \oplus \mathfrak{z}$ such that $f(x, t) = f_0(|x|, t)$ for every (x, t) in N, where $|x|^2 = \langle x, x\rangle_{\mathfrak{n}}$. Note that, when $m = 1$, the group N is the Heisenberg group $\mathbb{H}_n$ of the previous sections, and a function is $\mathfrak{v}$-radial if and only if it is $\mathrm{U}(n)$-invariant.

We recall some facts from [3,10] regarding the Gelfand spectrum Σ_{rad} of the commutative algebra of $\mathfrak{v}$-radial integrable functions. The algebra of left-invariant $\mathfrak{v}$-radial vector fields is generated by $L, i^{-1}T_1, \dots, i^{-1}T_m$, where L is the subLaplacian on N, i.e.,

$$L = -\sum_{j=1}^{2n} X_j^2.$$

The bounded $\mathfrak{v}$-spherical functions are given by the rules

$$\phi_{\lambda,d}^{\mathrm{rad}}(x, t) = e^{-\frac{1}{4}|\lambda||x|^2} \frac{\Lambda_d^{n-1}(\frac{1}{2}|\lambda||x|^2)}{\binom{d+n-1}{d}} e^{i\langle\lambda,t\rangle_{\mathfrak{n}}},$$

$$\phi_{\xi}^{\mathrm{rad}}(x, t) = \frac{(n-1)!}{(\sqrt{\xi}|x|/2)^{n-1}} J_{n-1}\big(\sqrt{\xi}|x|\big), \quad \forall (x, t) \in N$$

for every $\lambda \in \mathfrak{z} \setminus \{0\}$, $d \in \mathbb{N}$, $\xi \geqslant 0$. Moreover,

$$i^{-1} T_j \phi^{\mathrm{rad}}_{\lambda,d} = \lambda_j \phi^{\mathrm{rad}}_{\lambda,d}, \qquad L\phi^{\mathrm{rad}}_{\lambda,d} = |\lambda|(2d+n)\phi^{\mathrm{rad}}_{\lambda,d},$$
$$i^{-1} T_j \phi^{\mathrm{rad}}_{\xi} = 0, \qquad L\phi^{\mathrm{rad}}_{\xi} = \xi \phi^{\mathrm{rad}}_{\xi}.$$

Therefore [5] the Gelfand spectrum is homeomorphic to the following subset of $\mathbb{R}^{m+1}$:

$$\Sigma_{\mathrm{rad}} \simeq \big\{\big(\lambda, |\lambda|(2d+n)\big)\colon\ d \in \mathbb{N},\ \lambda \in \mathbb{R}^m \setminus \{0\}\big\} \cup \big\{(0,\dots,0,\xi)\colon\ \xi \geqslant 0\big\}. \tag{5.1}$$

We define the Gelfand transform $\mathcal{F}_{\mathrm{rad}} f$ of a $\mathfrak{v}$-radial integrable function f on N by the rule

$$\mathcal{F}_{\mathrm{rad}} f(\phi) = \int_N f(n)\overline{\phi(n)}\, dn \quad \forall \phi \in \Sigma_{\mathrm{rad}}.$$

According to the identification of Σ_{rad} with the subset of $\mathbb{R}^{m+1}$ given by (5.1), when λ is in $\mathbb{R}^m \setminus \{0\}$, d in $\mathbb{N}$, $\xi \geqslant 0$, we will more simply write

$$\widehat{f}\big(\lambda, |\lambda|(2d+n)\big) \quad \text{and} \quad \widehat{f}(0,\xi)$$

instead of $\mathcal{F}_{\mathrm{rad}} f(\phi^{\mathrm{rad}}_{\lambda,d})$ and $\mathcal{F}_{\mathrm{rad}} f(\phi^{\mathrm{rad}}_{\xi})$, respectively. As usual, $\mathcal{S}_{\mathrm{rad}}(N)$ will denote the space of $\mathfrak{v}$-radial Schwartz functions. The following result holds.

Theorem 5.1. *The Gelfand transform $\mathcal{F}_{\mathrm{rad}}$ is a topological isomorphism between $\mathcal{S}_{\mathrm{rad}}(N)$ and $\mathcal{S}(\Sigma_{\mathrm{rad}})$.*

The proof of Theorem 5.1 follows the same lines as in the previous sections and we only give a sketch of it. First of all, we need the analogue of Theorem 2.1 in the setting of H-type groups. It amounts to prove that if Φ is a Schwartz function on $\mathbb{R}^{m+1}$, then there exists f in $\mathcal{S}_{\mathrm{rad}}(N)$ such that $\widehat{f} = \Phi|_{\Sigma_{\mathrm{rad}}}$; moreover the correspondence $\Phi \mapsto f$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^{m+1})$ to $\mathcal{S}_{\mathrm{rad}}(N)$. These facts can be proved using [7] combined with the "tensor product" idea as in [15].

Our main interest is then in proving that the operator $\Phi \mapsto f$ is onto. To achieve this, we begin by proving the analogue of Theorem 2.2. This is done in Lemma 5.2. All other arguments apply easily; only the definition of the extended function of formula (3.1) is slightly different. Indeed, suppose that f is a $\mathfrak{v}$-radial Schwartz function on N. Then we define $E\widehat{f}$ on $\mathbb{R}^{m+1}$ by the rule

$$E\widehat{f}(\lambda,\xi) = \begin{cases} \sum_{d\in\mathbb{N}} \widehat{f}(\lambda, |\lambda|(2d+n))\varphi_1(\frac{\xi - |\lambda|(2d+n)}{|\lambda|}), & |\lambda| \neq 0,\ \xi \in \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.2 easily generalizes to this setting and one can conclude the proof reasoning as in Section 4 and applying Lemma 4.1 to the case where $n = 1$ and $m \geqslant 1$.

Lemma 5.2. *Let f be in $\mathcal{S}_{\mathrm{rad}}(N)$. Then for any multi-index* $\mathbf{j}$ *in* $\mathbb{N}^m$ *there exist functions* $f_{\mathbf{j}}$ *in* $\mathcal{S}_{\mathrm{rad}}(N)$ *such that for every* $M \geqslant 0$,

$$\widehat{f}\big(\lambda, |\lambda|(2d+n)\big) = \sum_{|\mathbf{j}| \leqslant M} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(0, |\lambda|(2d+n)\big) + \sum_{|\mathbf{j}|=M+1} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(\lambda, |\lambda|(2d+n)\big),$$

for every λ *in* $\mathbb{R}^m \setminus \{0\}$ *and for every nonnegative integer* d. *Moreover,* $f_{\mathbf{0}} = f$ *and for any multi-index* $\mathbf{j}$ *the map* $f \mapsto f_{\mathbf{j}}$ *is continuous on* $\mathcal{S}_{\mathrm{rad}}(N)$.

Proof. Let f be in $\mathcal{S}_{\mathrm{rad}}(N)$. Then $\widehat{f}(0,\cdot)$ is a Schwartz function on $\mathbb{R}^+$ and by [7] there exists a function h in $\mathcal{S}_{\mathrm{rad}}(N)$ such that

$$\widehat{h}\big(\lambda, |\lambda|(2d+n)\big) = \widehat{f}\big(\lambda, |\lambda|(2d+n)\big) - \widehat{f}\big(0, |\lambda|(2d+n)\big) \quad \forall \lambda \in \mathbb{R}^m \setminus \{0\},\ d \in \mathbb{N}.$$

Note that $\widehat{h}(0,\xi) = 0$ for every $\xi \geqslant 0$ and

$$\widehat{h}(0,\xi) = \mathcal{F}_{\mathfrak{v}} \mathcal{F}_{\mathfrak{z}} h(\sqrt{\xi} u, 0) \quad \forall u \in \mathfrak{v},\ |u| = 1,\ \xi \geqslant 0,$$

where $\mathcal{F}_{\mathfrak{v}}$ and $\mathcal{F}_{\mathfrak{z}}$ denote the Euclidean Fourier transforms with respect to the variables in $\mathfrak{v}$ and $\mathfrak{z}$, respectively. Therefore

$$\mathcal{F}_{\mathfrak{z}} h(x,0) = \int_{\mathfrak{z}} h(x,t)\, dt = 0 \quad \forall x \in \mathfrak{v},$$

and we can write, for every λ in $\mathbb{R}^m \setminus \{0\}$ and every x in $\mathfrak{v}$,

$$\mathcal{F}_{\mathfrak{z}} h(x,\lambda) = \int_0^1 \frac{d}{du}\big(\mathcal{F}_{\mathfrak{z}} h(x, u\lambda)\big)\, du = \sum_{j=1}^m \lambda_j \int_0^1 \big[\partial_{\lambda_j} \mathcal{F}_{\mathfrak{z}} h(x,\cdot)\big](u\lambda)\, du.$$

Let η be a smooth function on $\mathbb{R}^m$ such that $\eta(\lambda) = 0$ if $|\lambda| > 1$ and $\eta(\lambda) = 1$ if $|\lambda| < 1/4$. We can decompose the Schwartz function $\mathcal{F}_{\mathfrak{z}} h$ as

$$\begin{aligned} \mathcal{F}_{\mathfrak{z}} h(x,\lambda) &= \eta(\lambda) \mathcal{F}_{\mathfrak{z}} h(x,\lambda) + \big(1 - \eta(\lambda)\big) \mathcal{F}_{\mathfrak{z}} h(x,\lambda) \\ &= \sum_{j=1}^m \lambda_j \left[\eta(\lambda) \int_0^1 \big[\partial_{\lambda_j} \mathcal{F}_{\mathfrak{z}} h(x,\cdot)\big](u\lambda)\, du + \lambda_j \frac{1 - \eta(\lambda)}{|\lambda|^2} \mathcal{F}_{\mathfrak{z}} h(x,\lambda) \right]. \end{aligned}$$

Therefore, for every λ in $\mathbb{R}^m \setminus \{0\}$ and d in $\mathbb{N}$,

$$\begin{aligned} \widehat{h}\big(\lambda, |\lambda|(2d+n)\big) &= \int_{\mathfrak{v}} \mathcal{F}_{\mathfrak{z}} h(x,\lambda) \phi^{\mathrm{rad}}_{\lambda,d}(x,0)\, dx \\ &= \sum_{j=1}^m \lambda_j \int_{\mathfrak{v}} \mathcal{F}_{\mathfrak{z}}(\delta_j f)(x,\lambda) \phi^{\mathrm{rad}}_{\lambda,d}(x,0)\, dx \end{aligned}$$

$$= \sum_{j=1}^{m} \lambda_j \widehat{\delta_j f}\big(\lambda, |\lambda|(2d+n)\big),$$

where $\delta_j f$ is the function in $\mathcal{S}_{\mathrm{rad}}(N)$ defined by

$$\delta_j f(x,t) = \mathcal{F}_{\mathfrak{z}}^{-1}\left[\lambda \mapsto \eta(\lambda) \int_0^1 \big[\partial_{\lambda_j} \mathcal{F}_{\mathfrak{z}} h(x,\cdot)\big](u\lambda)\, du + \lambda_j \frac{1-\eta(\lambda)}{|\lambda|^2} \mathcal{F}_{\mathfrak{z}} h(x,\lambda)\right](t),$$

for every (x,t) in N. Then for every λ in $\mathbb{R}^m \setminus \{0\}$ and d in $\mathbb{N}$

$$\widehat{f}\big(\lambda, |\lambda|(2d+n)\big) = \widehat{f}\big(0, |\lambda|(2d+n)\big) + \sum_{j=1}^{m} \lambda_j \widehat{\delta_j f}\big(\lambda, |\lambda|(2d+n)\big).$$

Moreover any Schwartz norm of $\delta_j f$ can be controlled by a suitable Schwartz norm of h by the continuity properties of the Euclidean Fourier transform and by the properties of the cutoff function η. By [7], any Schwartz norm of h can be controlled by a suitable Schwartz norm of f. Let $\mathbf{e}_j$ denote the multi-index in $\mathbb{N}^m$ of length one whose entries are all trivial apart the jth, and define $f_{\mathbf{e}_j} = \delta_j f$. This concludes the proof in the case where $M = 0$.

We now proceed by induction on M. Suppose that the thesis holds true up to $M-1$, i.e., for every λ in $\mathbb{R}^m \setminus \{0\}$ and d in $\mathbb{N}$,

$$\widehat{f}\big(\lambda, |\lambda|(2d+n)\big) = \sum_{|\mathbf{j}| \leqslant M-1} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(0, |\lambda|(2d+n)\big) + \sum_{|\mathbf{j}|=M} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(\lambda, |\lambda|(2d+n)\big).$$

Then, reasoning as before on the functions $f_{\mathbf{j}}$ with $|\mathbf{j}| = M$, we obtain

$$\widehat{f}\big(\lambda, |\lambda|(2d+n)\big) = \sum_{|\mathbf{j}| \leqslant M} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(0, |\lambda|(2d+n)\big) + \sum_{|\mathbf{j}|=M} \sum_{j=1}^{m} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \lambda_j \widehat{\delta_j (f_{\mathbf{j}})}\big(\lambda, |\lambda|(2d+n)\big).$$

The last sum can be rewritten as

$$\sum_{|\mathbf{j}|=M+1} \frac{\lambda^{\mathbf{j}}}{\mathbf{j}!} \widehat{f_{\mathbf{j}}}\big(\lambda, |\lambda|(2d+n)\big),$$

where, if $|\mathbf{j}| = M+1$, the function $f_{\mathbf{j}}$ is defined by

$$\widehat{f_{\mathbf{j}}}\big(\lambda, |\lambda|(2d+n)\big) = \mathbf{j}! \sum_{\mathbf{p}+\mathbf{e}_j=\mathbf{j}} \frac{1}{\mathbf{p}!} \widehat{\delta_j (f_{\mathbf{p}})}\big(\lambda, |\lambda|(2d+n)\big),$$

for every λ in $\mathbb{R}^m \setminus \{0\}$ and d in $\mathbb{N}$. $\square$

References

[1] C. Benson, J. Jenkins, G. Ratcliff, T. Worku, Spectra for Gelfand pairs associated with the Heisenberg group, Colloq. Math. 71 (1996) 305–328.
[2] C. Benson, J. Jenkins, G. Ratcliff, The spherical transform of a Schwartz function on the Heisenberg group, J. Funct. Anal. 154 (1998) 379–423.
[3] E. Damek, F. Ricci, Harmonic analysis on solvable extensions of H-type groups, J. Geom. Anal. 2 (1992) 213–248.
[4] J. Faraut, K. Harzallah, Deux cours d'analyse harmonique, Progr. Math., vol. 69, Birkhäuser Boston, Boston, MA, 1987.
[5] F. Ferrari Ruffino, The topology of the spectrum for Gelfand pairs on Lie groups, Boll. Unione Mat. Ital., in press.
[6] D. Geller, Fourier analysis on the Heisenberg group. I. Schwartz space, J. Funct. Anal. 36 (1980) 205–254.
[7] A. Hulanicki, A functional calculus for Rockland operators on nilpotent Lie groups, Studia Math. 78 (1984) 253–266.
[8] A. Hulanicki, F. Ricci, A Tauberian theorem and tangential convergence for bounded harmonic functions on balls in $\mathbb{C}^n$, Invent. Math. 62 (1980) 325–331.
[9] A. Kaplan, Fundamental solution for a class of hypoelliptic PDE generated by composition of quadratic forms, Trans. Amer. Math. Soc. 258 (1980) 147–153.
[10] A. Korányi, Some applications of Gelfand pairs in classical analysis, in: Harmonic Analysis and Group Representations, C.I.M.E., Liguori, Napoli, 1980, pp. 333–348.
[11] B. Malgrange, Ideals of Differentiable Functions, Oxford Univ. Press, Bombay, 1966.
[12] G. Mauceri, Maximal operators and Riesz means on stratified groups, in: Proceedings of the Conference, Cortona, 1984, in: Sympos. Math., vol. XXIX, Academic Press, New York, 1987, pp. 47–62.
[13] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.
[14] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.
[15] A. Veneruso, Schwartz kernels on the Heisenberg group, Boll. Unione Mat. Ital. Sez. B 6 (2003) 657–666.
[16] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934) 63–89.